

Infrared behavior of the dispersion relations in high-temperature scalar QED

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ABSTRACT: We investigate the infrared properties of the next-to-leading-order dispersion relations in scalar quantum electrodynamics at high temperature in the context of hard-thermal-loop perturbation theory. Specifically, we determine the damping rate and the energy for scalars with ultrasoft momenta. We show by explicit calculations that an early external-momentum expansion, before the Matsubara sum is performed, gives exactly the same result as a late one. The damping rate is obtained up to fourth order included in the ultrasoft momentum and the energy up to second order. The damping rate is found sensitive in the infrared whereas the energy is not.

KEYWORDS: NLO Computations, Thermal Field Theory.

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1. Introduction

Because of asymptotic freedom, one thinks that the structure and properties of high-temperature quantum chromodynamics (QCD) should be accessible by perturbative means, however these are organized [1]. Originally, the standard loop-expansion [2, 3] ran into difficulties when attempting to describe the properties of slow-moving particles: gauge invariance was lost [4], a consequence of the fact that the standard loop-expansion did not reflect anymore a systematic expansion in powers of the small coupling constant. A re-organization of lowest-order terms in the correlation functions was therefore necessary and indeed performed: the so-called hard thermal loops (HTL) were systematically extracted from one-loop order diagrams and added to the lowest-order quantities [5–12]. It should also be mentioned that the need for resummation of standard loop-expansion is not peculiar to non-abelian theories but is also necessary in the abelian case, for quantum electrodynamics (QED) [12] and scalar QED [13], as well as for the simpler φ^4 -theory [14]. Thus are formed dressed propagators and vertices and a convenient description of the slow-moving quasi-particles becomes possible. Next-to-leading order quantities, in this new framework, are obtained via one-loop diagrams involving these dressed propagators and vertices. In this new context, some of the old difficulties are cured but others persist, most notably sensitivity to the infrared and the light-cone, essentially due to the fact that static

(chromo)magnetic fields remain unshielded even after the HTL reorganization. In QCD, these are thought of becoming screened at the so-called magnetic scale, which is believed to manifest itself non-perturbatively in the dispersion relations at next-to-leading order. Because of this, perturbation is believed to break down at some order, which depends on the quantity under consideration [15–17].

Still, useful information can be extracted from the perturbative regime. Indeed, using this method, an early explicit calculation of the zero-momentum transverse gluon damping rate shows that it is finite and positive [18]. A similar computation for the quarks has also been carried out independently in [19] and [20] and it yields a finite and positive number too. As for the calculation of static quantities such as free energies and screening masses, these are most conveniently carried out in the imaginary-time formalism since analytic continuation to real time is not necessary [21] anymore. For such quantities, there exists a simplified resummation scheme [22], based on the fact that in Euclidean formalism, the momentum can be soft only for the zero mode in the Matsubara sum. Hence dressing is necessary only for the static propagators, and calculations based on this are relatively simpler to carry out than those of the full resummation, necessary when considering dynamic quantities. The ‘reduced’ approach has been used in many instances like in the study of phase transitions in gauge theories [22], the calculation of the free energy in φ^4 -theory and QCD [23, 24] as well as in the computation of the electric screening mass in QED and scalar QED [21]. Other important quantities in this regard are the next-to-leading order Debye screening length [25] and next-to-leading order correction to the gluonic plasma frequency [26].

But as mentioned, difficulties still persist after HTL reorganization. For example, logarithmic sensitivity is encountered in an early estimation of the damping rate for a heavy fermion [8], and more generally for fast-moving particles [27–30]. Also, an estimation of the damping of moving quasi-particles is logarithmically dependent on the coupling [31, 32]. Of particular importance in this regard is the work [33, 34] where it is emphasized that the electron lifetime in the context of QED at finite temperature is plagued with infrared divergences, even after taking into account screening effects, and this is due to the exchange of unscreened ultrasoft magnetic photons with a contribution enhanced by the Bose-Einstein distribution. It is shown that a finite-temperature generalization of the Bloch-Nordsieck model allows the summation of the leading singular diagrams to all orders to obtain a long-time behavior for the electron propagator of the form $\exp\left[-\frac{\tilde{e}^2 T}{4\pi} t \ln(\tilde{e} T t/3)\right]$ instead of the generally assumed standard simple-pole behavior $\exp(-\gamma t)$, where \tilde{e} is the QED coupling constant, T the temperature and γ the damping rate. This clearly indicates the complexity of the analytic structure of the theory.

The infrared problem was also emphasized when attempting to estimate the damping rates of non-moving longitudinal gluons [35, 36] and slow-moving quarks [37, 38]. As just hinted to, what is of concern here is, partly, the analytic properties of QCD at finite temperature in the infrared. In order to investigate some of these properties, the context of scalar QED is useful because it offers a simpler setting in which calculations are facilitated by the fact that there are no hard thermal loops in the vertices and those related to

the scalar self-energy are momentum independent [13]. This situation allows sometimes carrying out almost complete momentum-dependent calculations. Of particular interest to us is an issue raised by the works [35, 36]: in order to carry forward analytically, one had to expand in powers of the external momentum p , considered ultrasoft to ensure the validity of the expansion, to allow one to perform intermediary solid-angle integrals which would have been, otherwise, intractable analytically. The particular feature of this expansion is that it was done in the imaginary-time formalism, before the Matsubara sum was performed and the analytic continuation to real energies taken. The subsequent appearance of infrared sensitivity in the final coefficients may then be linked to this ‘early expansion’. This is more significant if we know that the damping rate for non-moving longitudinal gluons has to be equal to the corresponding finite one for transverse gluons [39].

To tackle this specific issue in more depth, we embark in this work on the investigation of the next-to-leading order dispersion relation for scalars with ultrasoft momentum p in the context of next-to-leading order HTL perturbation of scalar QED. We will examine both the damping rate $\gamma_s(p)$ and the energy $\omega_s(p)$. Note that the damping rate $\gamma_s(p)$ has already been investigated in [40] and found, to second order in p , logarithmically infrared sensitive. In this work, the expansion of $\gamma_s(p)$ is pushed to fourth order in p and that of $\omega_s(p)$ to second order. What is interesting is that we perform the expansions by two different methods: (i) We delay the momentum expansion until after the Matsubara sum is performed and the analytic continuation to real energies taken. As for the damping rate $\gamma_s(p)$, we will find the coefficient of second order logarithmically sensitive to the infrared cut-off η as in [40] whereas the coefficient of fourth order will behave like $1/\eta^2$ (odd-order coefficients cancel). As for the energy $\omega_s(p)$, no infrared sensitivity will appear. (ii) We perform the early-momentum expansion and carry on with steps similar to [35, 36]. We find exactly the same results as those of the previous method. This must alleviate some pressure on the early-momentum expansion method: it may not be responsible for the appearance of infrared sensitivity. In the context of QCD, there is simply little hope of obtaining compact analytic expressions, and expansions such as the early-momentum one are sometimes necessary.

This article is organized as follows. After this introduction, the second section recapitulates the hard thermal loops and the dressing of the propagators. Section three discusses the calculation of the scalar damping rate $\gamma_s(p)$ by the two methods and section four the calculation of the scalar energy $\omega_s(p)$. Section five is devoted to the discussion of the results and includes some concluding remarks. An appendix is added and includes technical details from section three for which there is no need to display there.

2. Dressing the propagators and the dispersion relation

Let us first recapitulate the results giving the propagators and vertices to be used in the forthcoming next-to-leading order calculations. We use the imaginary-time formalism in which the four-momentum is $P^\mu = (p_0, \mathbf{p})$ such that $P^2 = p_0^2 + p^2$ with the scalar Matsubara frequency $p_0 = 2\pi nT$, n an integer and T the temperature. After the evaluation

of all frequency sums, p_0 is analytically continued to the real external energy ω using the analytic continuation $p_0 = -i\omega + 0^+$.

2.1 Dressed propagators

To leading order, the self-energies are obtained from undressed one-loop diagrams. For soft external momenta, that is for ω and p of order eT where e is the coupling constant, the dominant contributions to these diagrams come from loop momenta of the order of T , the hard scale. These contributions are called hard thermal loops. For the scalar self-energy, the hard thermal loop is given by [13]:

$$\delta\Sigma = -m_s^2, \quad (2.1)$$

with $m_s = eT/2$, the scalar thermal mass. Note that $\delta\Sigma$ is momentum independent. When dressed with the hard thermal loop, the leading-order propagator for the scalar becomes:

$${}^*\Delta_s(P) = \frac{1}{P^2 + m_s^2}. \quad (2.2)$$

The hard thermal loop $\delta\Pi^{\mu\nu}$ in the photon self-energy is given in [40] and can be expressed in terms of two independent scalar functions $\delta\Pi_l(K)$ and $\delta\Pi_t(K)$ given by:

$$\delta\Pi_l(K) = 3m_p^2 Q_1\left(\frac{ik_0}{k}\right); \quad \delta\Pi_t(K) = \frac{3}{5}m_p^2 \left[Q_3\left(\frac{ik_0}{k}\right) - Q_1\left(\frac{ik_0}{k}\right) - \frac{5}{3} \right], \quad (2.3)$$

where $m_p = eT/3$ is the photon thermal mass and the Q_i 's are Legendre functions of the second kind. In the strict Coulomb gauge, the components of the dressed photon propagator ${}^*\Delta_{\mu\nu}(K)$ are as follows:

$$\begin{aligned} {}^*\Delta_{00}(K) &= {}^*\Delta_l(K), & {}^*\Delta_{0i}(K) &= 0; \\ {}^*\Delta_{ij}(K) &= \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) {}^*\Delta_t(K), \end{aligned} \quad (2.4)$$

where ${}^*\Delta_l$ and ${}^*\Delta_t$ are the propagators for the longitudinal and transverse photons respectively. They have the following expressions:

$${}^*\Delta_l(K) = \frac{1}{k^2 - \delta\Pi_l(K)}; \quad {}^*\Delta_t(K) = \frac{1}{K^2 - \delta\Pi_t(K)}. \quad (2.5)$$

Now one peculiarity of scalar QED is that the vertices remain undressed, i.e., unaffected by the hard thermal loops [13]. The vertex with one photon and two scalar external lines (Q incoming, P outgoing) is:

$$\Gamma^\mu(P, Q) = -e(P + Q)^\mu, \quad (2.6)$$

and the vertex between two photons and two scalars is momentum independent and writes:

$$\Gamma^{\mu\nu} = 2e^2 \delta^{\mu\nu}. \quad (2.7)$$

2.2 Dispersion relation

The scalar damping rate $\gamma_s(p)$ and energy $\omega_s(p)$ are obtained from the scalar complex energy ω which satisfies the following full scalar dispersion relation [40]:

$$\omega^2 = p^2 - \Sigma(\omega, p), \tag{2.8}$$

where $\Sigma(\omega, p)$ is the full scalar self-energy. Taking into account the next-to-leading order terms in e and expanding everything around the leading order result $\omega_{s0}(p)$ (squared) for fixed p , we obtain:

$$\omega^2 = p^2 - \delta\Sigma(\omega_{s0}, p) - {}^*\Sigma(\omega_{s0}, p) - (\omega^2 - \omega_{s0}^2) \partial_{x^2} \delta\Sigma(x, p)|_{x=\omega_{s0}} + \mathcal{O}(e^4 T^2). \tag{2.9}$$

In this relation, $\omega_{s0}(p) = \sqrt{m_s^2 + p^2}$, pole of ${}^*\Delta_s(P)$ from (2.2) and solution to (2.8) to lowest order. $\delta\Sigma$ is nothing but the hard thermal loop given in (2.1) and ${}^*\Sigma$ is the next-to-leading order contribution. Using (2.1), we can rewrite (2.9) as:

$$\omega^2 = \omega_{s0}^2 - {}^*\Sigma(\omega_{s0}, p) + \mathcal{O}(e^4 T^2). \tag{2.10}$$

The full energy $\omega(p)$ is in general complex. If we denote its real part (the scalar energy) by $\omega_s(p)$, then we have:

$$\omega_s(p) = \omega_{s0} - \frac{\text{Re } \delta {}^*\Sigma(\omega_{s0}, p)}{2\omega_{s0}} + \mathcal{O}(e^3 T). \tag{2.11}$$

The prefix δ before ${}^*\Sigma$ in (2.11) indicates that the known leading hard-thermal-loop contribution has to be subtracted for ultraviolet convergence [13]. The damping rate for scalars is defined by $\gamma_s(p) = -\text{Im}\omega(p)$. It is e -times smaller than $\omega_{s0}(p)$, and so we have to lowest order [40]:

$$\gamma_s(p) = \frac{1}{2\omega_{s0}} \text{Im } {}^*\Sigma(\omega_{s0}, p) + \mathcal{O}(e^3 T). \tag{2.12}$$

We see that determining $\gamma_s(p)$ to lowest order in e and $\omega_s(p)$ to next-to-leading order amounts to calculating the imaginary and real parts of the next-to-leading order scalar self-energy ${}^*\Sigma$. From now on, we assume the scalar momentum p *ultrasoft*, i.e., of the order of $e^2 T$. Also, we will take $m_s \equiv 1$ in the sequel, to ease the notation. We will reintroduce it back in the final expressions.

3. Scalar damping rate

We start by calculating the damping rate $\gamma_s(p)$. The self-energy ${}^*\Sigma$ is the sum of two contributions [40]:

$${}^*\Sigma = {}^*\Sigma_1 + {}^*\Sigma_2, \tag{3.1}$$

where the contribution ${}^*\Sigma_1$, shown in figure 1 [41], is a one-loop dressed diagram with two one-photon-two-scalar vertices given in (2.6):

$${}^*\Sigma_1(P) = \text{Tr}_{\text{soft}} [\Gamma^\mu(P, Q) {}^*\Delta_s(Q) \Gamma^\nu(Q, P) {}^*\Delta_{\mu\nu}(K)], \tag{3.2}$$

and the contribution ${}^*\Sigma_2$, shown in figure 2, is the dressed tadpole diagram with a two-

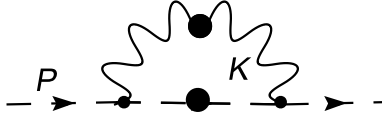


Figure 1: HTL-dressed one-loop-order scalar self-energy ${}^*\Sigma_1$.

photon-two-scalar vertex given in (2.7):

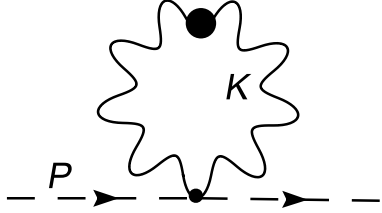


Figure 2: Next-to-leading order HTL-dressed scalar self-energy ${}^*\Sigma_2$.

$${}^*\Sigma_2 = \text{Tr}_{\text{soft}} [\Gamma^{\mu\nu} {}^*\Delta_{\mu\nu}(K)]. \quad (3.3)$$

In the above two relations, K is the internal photon loop-momentum, $Q = P - K$ and $\text{Tr} \equiv T \sum_{k_0} \int d^3k / (2\pi)^3$ with $k_0 = 2n\pi T$. The subscript ‘soft’ indicates that only soft momenta are allowed in the integral. To take account of potential infrared sensitivity, we will introduce an infrared cutoff η in the k -integration.

It turns out that ${}^*\Sigma_2$ is real. This is because there is only one dressed propagator involved in its expression (3.3) and the vertex there is undressed. It will also turn out to be momentum independent, see (4.6). Therefore, ${}^*\Sigma_2$ does not contribute to the damping rate $\gamma_s(p)$; it will only shift the mass m_s of the scalar. We thus can simply write:

$$\gamma_s(p) = \frac{1}{2\omega_{s0}} \text{Im} {}^*\Sigma_1(\omega_{s0}, p) + \mathcal{O}(e^3 T), \quad (3.4)$$

and henceforth, we will not manipulate ${}^*\Sigma_2$ until we reach the calculation of the scalar energy $\omega_s(p)$ in section four.

3.1 Late momentum expansion

In this subsection, we will first perform the Matsubara sums and analytically continue to real energies before expanding in powers of p . Using the structure (2.4) of the photon propagator in the strict Coulomb gauge and the expression of the one-photon-two-scalar vertex (2.6), we see that ${}^*\Sigma_1(P)$ is composed of two terms. The first term involves longitudinal photons and is denoted consequently by ${}^*\Sigma_{1l}(P)$, and the second one involves transverse photons and is denoted by ${}^*\Sigma_{1t}(P)$. Let us look first at ${}^*\Sigma_{1l}(P)$. We have:

$${}^*\Sigma_{1l}(P) = e^2 T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \left[(2p_0 - k_0)^2 {}^*\Delta_l(K) {}^*\Delta_s(Q) \right]. \quad (3.5)$$

Writing explicitly the integral over the solid angle of $\hat{\mathbf{k}}$ in a reference frame where $\hat{\mathbf{p}}$ is the principal axis, we have:

$${}^*\Sigma_{1l}(P) = \frac{e^2}{4\pi^2} T \sum_{k_0} \int_{\eta}^{+\infty} dk k^2 \int_{-1}^{+1} dx \left[(2p_0 - k_0)^2 {}^*\Delta_l(k_0, k) {}^*\Delta_s(q_0, q) \right], \quad (3.6)$$

with $x = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$. Note that $q = \sqrt{k^2 - 2pkx + p^2}$, which makes the integral over x not feasible for the moment.

We want to perform the Matsubara sum over k_0 . For this, we use the spectral decomposition of the two dressed propagators. In general, we have [42]:

$$\Delta_i(k_0, k) = \int_0^{1/T} d\tau e^{ik_0\tau} \int_{-\infty}^{+\infty} d\omega \rho_i(\omega, k) (1 + n(\omega)) e^{-\omega\tau}, \quad (3.7)$$

where i stands for l, t or s and $n(\omega)$ is the Bose-Einstein distribution. The explicit expressions of the spectral densities ρ_i are displayed in (3.12)-(3.15) below. Making this replacement, the sum over k_0 can now be performed. The subsequent steps are standard: One imaginary-time integration is eliminated by a delta function and the second one yields an energy denominator. Everywhere except in the energy denominator p_0 is replaced by $2\pi nT$. The analytic continuation to real energies is taken at this stage and obtained by the replacement $ip_0 \rightarrow \omega_{s0}(p) + i0^+$. The imaginary part is extracted using the known relation $1/(x + i0^+) = \text{Pr}(1/x) - i\pi\delta(x)$. We thus have:

$$\text{Im } {}^*\Sigma_{1l}(P) = -\frac{e^2 T}{4\pi} \int_{\eta}^{+\infty} dk k^2 \int_{-1}^{+1} dx \int_{-\infty}^{+\infty} d\omega \frac{\omega_{s0}(2\omega_{s0} - \omega)^2}{\omega(\omega_{s0} - \omega)} \rho_l(\omega, k) \rho_s(\omega_{s0} - \omega, q). \quad (3.8)$$

In the above relation, only soft values of ω are to contribute in the integral, and so we have used the approximation $n(\omega) \simeq T/\omega$.

The transverse contribution ${}^*\Sigma_{1t}$ is handled in similar steps and we obtain for its imaginary part the following expression:

$$\text{Im } {}^*\Sigma_{1t}(P) = \frac{e^2 T}{\pi} p^2 \int_{\eta}^{+\infty} dk k^2 \int_{-1}^{+1} dx (1 - x^2) \int_{-\infty}^{+\infty} d\omega \frac{\omega_{s0}}{\omega(\omega_{s0} - \omega)} \rho_t(\omega, k) \rho_s(\omega_{s0} - \omega, q). \quad (3.9)$$

Note that the transverse contribution to $\gamma_s(p)$ already starts at order p^2 .

Now dividing by $2\omega_{s0}(p)$ as required in (3.4), we get the following longitudinal and transverse contributions to the damping rate:

$$\begin{aligned} \gamma_{sl}(p) &= -\frac{e^2 T}{8\pi} \int_{\eta}^{+\infty} dk k^2 \int_{-1}^{+1} dx \int_{-\infty}^{+\infty} d\omega \frac{(2\omega_{s0} - \omega)^2}{\omega(\omega_{s0} - \omega)} \rho_l(\omega, k) \rho_s(\omega_{s0} - \omega, q); \\ \gamma_{st}(p) &= \frac{e^2 T}{2\pi} p^2 \int_{\eta}^{+\infty} dk k^2 \int_{-1}^{+1} dx \int_{-\infty}^{+\infty} d\omega \frac{(1 - x^2)}{\omega(\omega_{s0} - \omega)} \rho_t(\omega, k) \rho_s(\omega_{s0} - \omega, q). \end{aligned} \quad (3.10)$$

According to (3.4), the damping rate itself will be:

$$\gamma_s(p) = \gamma_{sl}(p) + \gamma_{st}(p). \quad (3.11)$$

Next we move to perform the integrals involved in the expressions of (3.10) above. The spectral densities $\rho_{l,t}$ are known [42, 12]:

$$\rho_{l,t}(\omega, k) = \mathfrak{z}_{l,t}(k) [\delta(\omega - \omega_{l,t}(k)) - \delta(\omega + \omega_{l,t}(k))] + \beta_{l,t}(\omega, k) \Theta(k^2 - \omega^2), \quad (3.12)$$

where $\mathfrak{z}_{l,t}(k)$ are the residue functions and $\beta_{l,t}(\omega, k)$ the cut functions. The residue functions are given by:

$$\mathfrak{z}_l(k) = -\frac{\omega(\omega^2 - k^2)}{k^2(4/3 - \omega^2 + k^2)} \Big|_{\omega=\omega_l(k)}; \quad \mathfrak{z}_t(k) = \frac{\omega(\omega^2 - k^2)}{4\omega^2/3 - (\omega^2 - k^2)^2} \Big|_{\omega=\omega_t(k)}, \quad (3.13)$$

and the cut functions have the following expressions:

$$\beta_l(k, \omega) = -\frac{2\omega}{3k} \left[\left(4/3 + k^2 - \frac{2\omega}{3k} \ln \frac{k+\omega}{k-\omega} \right)^2 + \frac{4\pi^2\omega^2}{9k^2} \right]^{-1}, \quad (3.14)$$

$$\beta_t(k, \omega) = \frac{\omega(k^2 - \omega^2)}{3k^3} \left[\left(k^2 - \omega^2 + \frac{2\omega^2}{3k^2} \left(1 + \frac{k^2 - \omega^2}{2k\omega} \ln \frac{k+\omega}{k-\omega} \right) \right)^2 + \left(\frac{\pi\omega(k^2 - \omega^2)}{3k^3} \right)^2 \right]^{-1}.$$

Remember that all quantities are in units of m_s . The spectral density ρ_s does not have a cut. It simply writes:

$$\rho_s(\omega, k) = \mathfrak{z}_s(k) [\delta(\omega - \omega_{s0}(k)) - \delta(\omega + \omega_{s0}(k))], \quad (3.15)$$

with $\mathfrak{z}_s(k) = 1/2\omega_{s0}(k)$.

We therefore replace the spectral densities by their respective expressions (3.12) and (3.15). The integration over ω disappears with the delta functions of (3.15). From the expression of $\rho_{l,t}$ in (3.12), there are going to be two kinds of contributions, i.e., a δ -contribution and a Θ -contribution. Because of kinematics, the δ -contribution is always zero whereas the Θ -contribution survives. The integrations over x and k have to be performed numerically. One would fit the behaviors of $\gamma_{sl}(p)$ and $\gamma_{st}(p)$ to several ultrasoft values of p , but in the spirit of the present work and in order to make direct comparison with the early momentum expansion method to be presented shortly, we display the results for $\gamma_{sl}(p)$ and $\gamma_{st}(p)$ in powers of p to fourth order. We obtain:

$$\begin{aligned} \gamma_{sl}(p) &= \frac{e^2 T}{16\pi} [1.44253 + 0.309278 \bar{p}^2 - 0.133949 \bar{p}^4 + \mathcal{O}(\bar{p}^6)] + \mathcal{O}(e^3 T); \\ \gamma_{st}(p) &= \frac{e^2 T}{4\pi} \left[- (1.65937 + 1.62114 \ln \bar{\eta}) \bar{p}^2 \right. \\ &\quad \left. + \left(71.3264 + 57.4152 \ln \bar{\eta} + \frac{1.94537}{\bar{\eta}^2} \right) \bar{p}^4 + \mathcal{O}(\bar{p}^6) \right] + \mathcal{O}(e^3 T). \end{aligned} \quad (3.16)$$

The thermal mass m_s has been reintroduced and here, $\bar{p} = p/m_s$ and $\bar{\eta} = \eta/m_s$.

To order \bar{p}^2 , these are the results obtained already in [40]. Note that the longitudinal contribution $\gamma_{sl}(p)$ is safe from any infrared sensitivity whereas the transverse contribution $\gamma_{st}(p)$ is infrared sensitive, this despite the fact that both contributions have been handled

in exactly similar ways. The reason why we have wanted to push the expansion to order \bar{p}^4 is to show that other forms of infrared sensitivity may appear, other than the familiar $\ln \bar{\eta}$. Indeed, we clearly see the power-like behavior $1/\bar{\eta}^2$ in the \bar{p}^4 -coefficient of $\gamma_{st}(p)$. These issues will be furthered in section five. According to (3.11), the scalar damping rate is:

$$\gamma_s(p) = \frac{e^2 T}{16\pi} \left[1.44253 - (6.32820 + 6.48456 \ln \bar{\eta}) \bar{p}^2 + \left(285.172 + 229.661 \ln \bar{\eta} + \frac{7.78148}{\bar{\eta}^2} \right) \bar{p}^4 + \mathcal{O}(\bar{p}^6) \right] + \mathcal{O}(e^3 T). \quad (3.17)$$

3.2 Early momentum expansion

Now let us perform the same calculation while introducing from the start an early momentum expansion, before the Matsubara sum and analytic continuation to real energies are done. First we perform analytically the integrals over the solid angle of $\hat{\mathbf{k}}$ in the same reference frame where $\hat{\mathbf{p}}$ is the principal axis. The difficulty comes from the presence of functions of $q = \sqrt{k^2 - 2pkx + p^2}$, mainly the scalar dressed propagator ${}^* \Delta_s(q_0, q)$. These need to be expanded and, in order to do this, we use the following expansion to fourth order in the external momentum p :

$$\begin{aligned} {}^* \Delta_s(q_0, q) = & \left[1 - px \partial_k + \frac{p^2}{2} \left[\frac{1-x^2}{k} \partial_k + x^2 \partial_k^2 \right] \right. \\ & + \frac{p^3 x}{2} \left[\frac{1-x^2}{k^2} \partial_k - \frac{1-x^2}{k} \partial_k^2 - \frac{x^3}{3} \partial_k^3 \right] + \frac{p^4}{4k} \left[\frac{-(1-6x^2+5x^4)}{2k^2} \partial_k \right. \\ & \left. \left. + \frac{(1-6x^2+5x^4)}{2k} \partial_k^2 + 2x^2(1-x^2) \partial_k^3 + \frac{x^4 k}{6} \partial_k^4 \right] + \mathcal{O}(p^5) \right] {}^* \Delta_s(q_0, k). \quad (3.18) \end{aligned}$$

Here ∂_k stands for $\partial/\partial k$ and, remember, $x = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$. Consider first ${}^* \Sigma_{1l}(P)$. Insert the above expansion in its expression (3.5) and then perform the integrations over x which become straightforward. All odd orders in p cancel out and we are left with:

$$\begin{aligned} {}^* \Sigma_{1l}(P) = & \frac{e^2}{2\pi^2} T \sum_{k_0} \int_{\eta}^{+\infty} dk k^2 (2p_0 - k_0)^2 {}^* \Delta_l(k_0, k) \\ & \times \left[1 + \frac{p^2}{3} \left(\frac{1}{k} \partial_k + \frac{1}{2} \partial_k^2 \right) + \frac{p^4}{30} \left(\partial_k^3 + \frac{k}{4} \partial_k^4 \right) + \mathcal{O}(p^6) \right] {}^* \Delta_s(q_0, k). \quad (3.19) \end{aligned}$$

Only at this stage the Matsubara sum and analytic continuation to real energies are done. Here too we replace the dressed propagators by their spectral decompositions given in (3.7) and follow the standard steps sketched right after (3.7) in order to extract the imaginary part. Dividing by $2\omega_{s0}(p)$ as indicated in (2.12), we obtain the following expression:

$$\begin{aligned} \gamma_{sl}(p) = & - \frac{e^2 T}{4\pi} \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} d\omega \frac{(2\omega_{s0} - \omega)^2}{\omega(\omega_{s0} - \omega)} \rho_l(\omega, k) \\ & \times \left[1 + \frac{p^2}{3} \left(\frac{1}{k} \partial_k + \frac{1}{2} \partial_k^2 \right) + \frac{p^4}{30} \left(\partial_k^3 + \frac{k}{4} \partial_k^4 \right) + \mathcal{O}(p^6) \right] \rho_s(\omega_{s0} - \omega, k). \quad (3.20) \end{aligned}$$

The transverse-photon contribution ${}^*\Sigma_{1t}(P)$ is handled in similar steps. Dividing by $2\omega_{s0}(p)$, we obtain from it the following transverse contribution to the damping rate:

$$\begin{aligned} \gamma_{st}(p) &= \frac{2e^2T}{3\pi} p^2 \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} \frac{d\omega}{\omega(\omega_{s0} - \omega)} \rho_t(\omega, k) \\ &\times \left[1 + \frac{2p^2}{5} \left(\frac{1}{k} \partial_k + \frac{1}{4} \partial_k^2 \right) + \mathcal{O}(p^6) \right] \rho_s(\omega_{s0} - \omega, k). \end{aligned} \quad (3.21)$$

As one sees, there are different types of terms involved in (3.20) and (3.21). In the sequel, we will briefly show how we carry through with the simplest of such terms and relegate the others to the appendix.

The simplest of the integrals we have to handle is the one that involves a $\rho\rho$ contribution with no derivatives. Generically, we consider an integral of the type:

$$I_{\rho\rho} = \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \rho_i(\omega, k) \rho_s(1 - \omega, k), \quad (3.22)$$

where i stands for l (longitudinal) or t (transverse). According to the form of the spectral functions (3.12) and (3.15), there are two kinds of contributions: a $\delta\delta$ (residue-residue) contribution and a $\Theta\delta$ (residue-cut) contribution. The $\delta\delta$ contribution requires that the energies satisfy $\pm\omega_{s0}(k) \pm\omega_i(k) = 1$, four constraints which are always forbidden by the dispersion relations. Hence the $\delta\delta$ contribution is always zero because of kinematics. The $\Theta\delta$ contribution writes

$$\int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \mathfrak{z}_s(k) \beta_i(k, \omega) [\delta(1 - \omega - \omega_{s0}) - \delta(1 - \omega + \omega_{s0})] \Theta(k - |\omega|).$$

A non-zero contribution must satisfy $\omega = 1 \pm \omega_{s0}$, together with $-k \leq \omega \leq k$. It is not difficult to see that only the case $\omega = 1 - \omega_{s0}(k)$ is allowed, and this for all values $k \geq \eta$. The integration over ω is straightforward and we obtain for this type of integral:

$$I_{\rho\rho} = \int_{\eta}^{+\infty} dk \mathfrak{z}_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_i(k, 1 - \omega_{s0}). \quad (3.23)$$

Note that only integrals of the type $I_{\rho\rho}$ contribute to the coefficient of lowest order in p in the scalar damping rate. As already mentioned, the other terms are presented in the appendix. They are worked out along similar lines, with more care when derivatives of distributions are present.

At the present stage, we are ready to evaluate the integrals. Let us work out a specific, general enough, example in some detail. Let us take it from the coefficient of p^4 in the expression in (3.21) for $\gamma_{st}(p)$, namely the $\rho_t(\omega, k) \partial_k \rho_s(1 - \omega, k)$ contribution. The corresponding function f is $f(k, \omega, \omega') = k/(\omega\omega')$. The integral involved is of the type $I_{\rho\partial_k\rho}$, see (A.2). Let us look at the first term $\mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0})$ where ω_{s0} stands here for $\omega_{s0}(k)$. Using the expressions of f , the residue function $\mathfrak{z}_s(k)$ given after (3.15) and $\beta_t(k, 1 - \omega_{s0})$ given in (3.14), we perform a small- k expansion of the product to obtain the following small- k behavior:

$$\mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0}) = -0.607927/k + 4.70205k + 30.0885k^3 + \mathcal{O}(k^5). \quad (3.24)$$

It is clear that this is going to lead to a logarithmic divergence. To extract this divergence from the integral, we split the latter into two parts: one integration from η to any finite number plus one second integration from that finite number to $+\infty$. The integral writes then:

$$\begin{aligned}
 \int_{\eta}^{+\infty} dk \mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0}) &= 0.607927 \ln \eta - 0.607927 \ln k_{s0}(\ell) \\
 &+ \int_0^{k_{s0}(\ell)} dk [\mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0}) + 0.607927/k] \\
 &+ \int_{\ell}^1 dx \mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0}) \Big|_{k=k_{s0}(x)}. \quad (3.25)
 \end{aligned}$$

What we have done is simple. In the finite part (the third term in (3.25)), we have changed the integration variable from k to $x \equiv k/\omega_{s0}(k)$, which implies that $k \equiv k_{s0}(x)$. Note that $k \rightarrow \infty$ implies $x \rightarrow 1$. The finite value in question that splits the original integral into two parts is then chosen in terms of x instead of k and is denoted by ℓ . The second term in (3.25) is the original divergent piece of the total integral from the integrand of which we have subtracted $-0.607927/k$ in order to render it safe in the infrared, and hence the lower bound η is replaced by 0. We must then of course add the contribution $-0.607927 \int_{\eta}^{k_{s0}(\ell)} \frac{dk}{k} = 0.607927 \ln \eta - 0.607927 \ln k_{s0}(\ell)$. As already mentioned, the finite value ℓ is arbitrary, but practically it must be chosen small enough in order to make the integral $\int_0^{k_{s0}(\ell)} dk [\mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0}) + 0.607927/k]$ numerically feasible. Indeed, though we are assured of its finiteness *analytically*, both integrands $\mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0})$ and $0.607927/k$ are still each (here logarithmically) divergent for small k . However, when ℓ is small, we can use a small- k expansion in order to get a number for the integral. We have pushed the expansion to $\mathcal{O}(k^{11})$ and we start having good convergence for already $\ell = 0.6$. Also, we must (and do) check that the final result does not depend on a particular value of ℓ . We finally get:

$$\int_{\eta}^{+\infty} dk \mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_t(k, 1 - \omega_{s0}) = 0.607927 \ln \eta + 0.702549. \quad (3.26)$$

Now to the second contribution to the example we have chosen to detail, which is $\int_{\eta}^{+\infty} dk \omega'_{s0} \mathfrak{z}_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)]|_{\omega=\omega_{s0}}$. A small- k expansion of the integrand yields:

$$\omega'_{s0} \mathfrak{z}_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_t(k, 1 - \omega)]|_{\omega=\omega_{s0}} = -2.43171/k^3 + 24.8687/k - 208.701k + \mathcal{O}(k^3). \quad (3.27)$$

Here we have a $1/\eta^2$ behavior in addition to the familiar $\ln \eta$. We therefore write:

$$\begin{aligned}
 &\int_{\eta}^{+\infty} dk \omega'_{s0} \mathfrak{z}_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_t(k, 1 - \omega)]|_{\omega=\omega_{s0}} \\
 &= -1.2159/\eta^2 + 1.2159/k_s(\ell)^2 - 24.8687 \ln \eta + 24.8687 \ln k_s(\ell) \\
 &\quad + \int_0^{k_{s0}(\ell)} dk \left[\omega'_{s0} \mathfrak{z}_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_t(k, 1 - \omega)]|_{\omega=\omega_{s0}} + \frac{2.43171}{k^3} - \frac{24.8687}{k} \right]
 \end{aligned}$$

$$+ \int_{\ell}^1 dx \omega'_{s0} \partial_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_t(k, 1 - \omega)]|_{k=k_{s0}(x)}. \quad (3.28)$$

We have proceeded as already explained. The integrations are smooth and no additional particular problem arises. Good convergence starts at $\ell = 0.6$ and the ℓ -independence of the sum is checked systematically. The final result is:

$$\int_{\eta}^{+\infty} dk \omega'_{s0} \partial_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_t(k, 1 - \omega)]|_{\omega=\omega_{s0}} = -1.2159/\eta^2 - 24.8687 \ln \eta - 27.2104. \quad (3.29)$$

Putting these two contributions together, we arrive for this specific example at:

$$I_{\rho \partial_k \rho}|_{\text{example}} = -1.2159/\eta^2 - 24.26077 \ln \eta - 26.507851. \quad (3.30)$$

All the terms are treated along similar steps: infrared divergences are systematically detected by a small- k expansion and extracted manually; the finite-part integrals performed numerically. We sum all the contributions and obtain exactly the two results (3.16) for $\gamma_{sl}(p)$ and $\gamma_{st}(p)$, namely:

$$\begin{aligned} \gamma_{sl}(p) &= \frac{e^2 T}{16\pi} [1.44253 + 0.309278 \bar{p}^2 - 0.133949 \bar{p}^4 + \mathcal{O}(\bar{p}^6)] + \mathcal{O}(e^3 T); \\ \gamma_{st}(p) &= \frac{e^2 T}{4\pi} \left[- (1.65937 + 1.62114 \ln \bar{\eta}) \bar{p}^2 \right. \\ &\quad \left. + \left(71.3264 + 57.4152 \ln \bar{\eta} + \frac{1.94537}{\bar{\eta}^2} \right) \bar{p}^4 + \mathcal{O}(\bar{p}^6) \right] + \mathcal{O}(e^3 T). \end{aligned} \quad (3.31)$$

Note that by this method too the longitudinal contribution $\gamma_{sl}(p)$ is free from any infrared divergence whereas the transverse contribution is infrared sensitive.

4. Scalar energy

Now we turn to the determination of the scalar energy $\omega_s(p)$ to next-to-leading order in the coupling constant e . It is defined in (2.11). We want also to obtain it in both the late and early p -expansion. We will see that in this case too the same result is obtained, free from any infrared divergence.

4.1 Late momentum expansion

Recall that the next-to-leading order self-energy ${}^* \Sigma$ is the sum of two diagrams ${}^* \Sigma_1$ and ${}^* \Sigma_2$, see (3.1), (3.2) and (3.3). Recall also that we have stated that ${}^* \Sigma_2$ is real. Indeed, using the structure of the two-photon-two-scalar vertex (2.7) and that of the photon propagator in the strict Coulomb gauge (2.4), we have:

$${}^* \Sigma_2 = 2e^2 T \sum_{k_0} \int \frac{d^3 k}{(2\pi)^3} [{}^* \Delta_l(K) + 2 {}^* \Delta_t(K)]. \quad (4.1)$$

The angular integrals over the k -solid angle and the sum over k_0 are done as explained before. No p -expansion is needed. We obtain the following expression:

$${}^*\Sigma_2 = \frac{e^2 T}{\pi^2} \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} [\rho_l(\omega, k) + 2\rho_t(\omega, k)], \quad (4.2)$$

as just restated, real. On the contrary, ${}^*\Sigma_1$ has an imaginary part which we have just manipulated in the previous section, both with late and early p -expansions. One useful way to obtain the real part of ${}^*\Sigma_1$ is to use its imaginary part and the following dispersion relation [13]:

$$\text{Re Tr}^* \Delta_{l,t}(k_0, k) {}^* \Delta_s(q_0, q) f(k) = \int_{-\infty}^{+\infty} \frac{dt}{t - \omega_{s0}} \frac{1}{\pi} [\text{Im Tr}^* \Delta_{l,t}(k_0, k) {}^* \Delta_s(q_0, q) f(k)]_{\omega_{s0}=t}. \quad (4.3)$$

Using this dispersion relation and performing the Matsubara sum before any momentum expansion, we obtain for the real part of the longitudinal contribution ${}^*\Sigma_{1l}$ the following expression:

$$\text{Re } {}^*\Sigma_{1l}(P) = -\frac{e^2 T}{4\pi^2} \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} \frac{dt}{t - \omega_{s0}} \int_{-1}^{+1} dx \int_{-\infty}^{+\infty} d\omega \frac{(2\omega_{s0} - \omega)^2 t}{\omega(t - \omega)} \rho_l(\omega, k) \rho_s(t - \omega, q). \quad (4.4)$$

Remember, $x = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$. For the transverse contribution ${}^*\Sigma_{1t}$, we obtain the following expression:

$$\begin{aligned} \text{Re } {}^*\Sigma_{1t}(P) &= \frac{e^2 T}{\pi^2} p^2 \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} \frac{dt}{t - \omega_{s0}} \int_{-1}^{+1} dx (1 - x^2) \\ &\times \int_{-\infty}^{+\infty} \frac{d\omega}{\omega(t - \omega)} t \rho_t(\omega, k) \rho_s(t - \omega, q). \end{aligned} \quad (4.5)$$

The next step is to perform the integrals in (4.2), (4.4) and (4.5). For ${}^*\Sigma_2$, we replace the photon dispersion relations by their expressions in (3.12). The residue part leaves one integration over k and the cut part restricts the integration over ω between $-k$ and k . Otherwise the integrations themselves are done numerically. No early or late momentum expansion is necessary and no special problem arises. In particular, the infrared behavior is safe. Recall that we have to subtract the leading order terms for ultraviolet convergence as indicated after (2.9). From the longitudinal photon we obtain $-0.183776 e^2 T$ and from the transverse photon $-0.000443967 e^2 T$, a much smaller contribution. Summing the two, we get:

$${}^*\Sigma_2 = -0.18422 e^2 T + \mathcal{O}(e^3 T). \quad (4.6)$$

No p -dependence as announced previously, which means ${}^*\Sigma_2$ will only correct the scalar thermal mass m_s , a correction free from any infrared divergence.

To perform the integrations in (4.4) and (4.5), we replace the scalar and photon spectral densities by their respective expressions (3.15) and (3.12). Because the scalar spectral density does not involve a cut, the integration over t is trivial. Here too we could give values to the external momentum p and fit the resulting curves, but in the spirit of the present work and in view of the coming comparison with the results from the early momentum

expansion method, we expand the integrand in powers of p and perform analytically the integration over x . The residue part in the photon spectral density will only leave the integration over k whereas the cut part will restrict the integration over ω . The remaining single and double integrals are done numerically. Here too no special problem arises and the infrared region is safe. After subtracting the corresponding leading-order terms for ultraviolet convergence, we obtain:

$$\begin{aligned} \text{Re } {}^*\Sigma_{1l}(P) &= e^2 T [-0.0746037 - 0.0201876 \bar{p}^2 + \mathcal{O}(\bar{p}^4)] + \mathcal{O}(e^3 T); \\ \text{Re } {}^*\Sigma_{1t}(P) &= e^2 T [-0.165908 \bar{p}^2 + \mathcal{O}(\bar{p}^4)] + \mathcal{O}(e^3 T). \end{aligned} \quad (4.7)$$

It remains to sum (4.6) and (4.7) and divide by $2\omega_{s0}(p)$ as instructed in (2.11) to obtain the scalar energy:

$$\omega_s(p) = m_s [(1 + 0.258824e + \mathcal{O}(e^2)) + (0.5 + 0.056684e + \mathcal{O}(e^2)) \bar{p}^2 + \mathcal{O}(\bar{p}^4)]. \quad (4.8)$$

4.2 Early momentum expansion

Finally, let us recalculate the scalar energy $\omega_s(p)$ while introducing the expansion in p at an early stage, before performing the Matsubara sum. Since ${}^*\Sigma_2$ does not depend on p , it will not be affected by this expansion; only ${}^*\Sigma_1$ is affected. As in the calculation of $\gamma_s(p)$, the integration over x becomes straightforward. We do the Matsubara sum and analytically continue to $\omega_{s0}(p)$. We obtain for the longitudinal contribution:

$$\begin{aligned} \text{Re } {}^*\Sigma_{1l}(P) &= -\frac{e^2 T}{2\pi^2} \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} \frac{dt}{t - \omega_{s0}} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega(t - \omega)} \rho_l(\omega, k) \\ &\quad \times t (2\omega_{s0} - \omega)^2 \left[1 + \frac{p^2}{3} \left(\frac{1}{k} \partial_k + \frac{1}{2} \partial_k^2 \right) + \dots \right] \rho_s(t - \omega, k). \end{aligned} \quad (4.9)$$

Odd powers in p cancel. The transverse contribution (4.5) is already of order p^2 and does not need further p -expansion. The x -integral done, Matsubara sum and analytical continuation to $\omega_{s0}(p)$ performed, we obtain:

$$\text{Re } {}^*\Sigma_{1t}(P) = \frac{4e^2 T}{3\pi^2} p^2 \int_{\eta}^{+\infty} dk k^2 \int_{-\infty}^{+\infty} \frac{dt}{t - \omega_{s0}} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega(t - \omega)} t \rho_t(\omega, k) \rho_s(t - \omega, k). \quad (4.10)$$

What remains now is to plug in the expressions of the spectral densities and perform the integrals over ω and t , and then over the momentum k . The scalar spectral density ρ_s has no cut part, which makes the integration over t simple. The residue part in $\rho_{l,t}$ eliminates the integration over ω and the cut part restricts its limits. Numerical work is needed for the rest. Here follows a display of generic terms. First we have the type:

$$\begin{aligned} R_{\rho\rho} &= \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega f(k, \omega, t) \rho_i(k, \omega) \rho_s(k, t - \omega) \\ &= \int_{\eta}^{+\infty} dk [\mathfrak{z}_i \mathfrak{z}_s [f(k, \omega_i, \omega_i + \omega_{s0}) - f(k, \omega_i, \omega_i - \omega_{s0}) - f(k, -\omega_i, -\omega_i + \omega_{s0}) \\ &\quad + f(k, -\omega_i, -\omega_i - \omega_{s0})] \end{aligned}$$

$$+ \int_{-k}^{+k} d\omega \mathfrak{z}_s \beta_i(k, \omega) [f(k, \omega, \omega + \omega_{s0}) - f(k, \omega, \omega - \omega_{s0})]. \quad (4.11)$$

The subscript i stands for l or t . Note that only integrals of type $R_{\rho\rho}$ contribute to the coefficient of zeroth order in p in the real part. The second type of integrals is the following:

$$\begin{aligned} R_{\rho\partial_k\rho} &= \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega f(k, \omega, t) \rho_i(k, \omega) \partial_k \rho_s(t - \omega, k) \\ &= \int_{\eta}^{+\infty} dk [\mathfrak{z}_i \mathfrak{z}'_s [f(k, \omega_i, \omega_i + \omega_{s0}) - f(k, \omega_i, \omega_i - \omega_{s0}) - f(k, -\omega_i, -\omega_i + \omega_{s0}) \\ &\quad + f(k, -\omega_i, -\omega_i - \omega_{s0})] + \omega'_{s0} \mathfrak{z}_i \mathfrak{z}_s [\partial_t (f(k, \omega_i, t) + f(k, -\omega_i, -t))|_{t=\omega_i+\omega_{s0}} \\ &\quad + \partial_t (f(k, \omega_i, t) + f(k, -\omega_i, t))|_{t=\omega_i-\omega_{s0}}] + \int_{-k}^{+k} d\omega [\mathfrak{z}'_s \beta_i(k, \omega) [f(k, \omega, \omega + \omega_{s0}) \\ &\quad - f(k, \omega, \omega - \omega_{s0})] + \omega'_{s0} \mathfrak{z}_s \beta_i(k, \omega) [\partial_t f(k, \omega, t)|_{t=\omega+\omega_{s0}} + \partial_t f(k, \omega, t)|_{t=\omega-\omega_{s0}}]]. \end{aligned} \quad (4.12)$$

The third and last type of integrals is the one involving a second-order derivative in k ; it writes:

$$\begin{aligned} R_{\rho\partial_k^2\rho} &= \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega f(k, \omega, t) \rho_i(k, \omega) \partial_k^2 \rho_s(t - \omega, k) \\ &= \int_{\eta}^{+\infty} dk [\mathfrak{z}_i (\mathfrak{z}''_s + \omega'^2_{s0} \mathfrak{z}_s \partial_t^2) [(f(k, \omega_i, t))|_{t=\omega_i+\omega_{s0}} - (f(k, \omega_i, t))|_{t=\omega_i-\omega_{s0}} \\ &\quad - (f(k, -\omega_i, t))|_{t=-\omega_i+\omega_{s0}} + (f(k, -\omega_i, t))|_{t=-\omega_i-\omega_{s0}}] + (2\omega'_{s0} \mathfrak{z}'_s + \omega''_{s0} \mathfrak{z}_s) \mathfrak{z}_i \\ &\quad \times [\partial_t (f(k, \omega_i, t) + f(k, -\omega_i, -t))|_{t=\omega_i+\omega_{s0}} + \partial_t (f(k, \omega_i, t) + f(k, -\omega_i, t))|_{t=\omega_i-\omega_{s0}}] \\ &\quad + \int_{-k}^{+k} d\omega [(\mathfrak{z}''_s + \mathfrak{z}_s \omega'^2_{s0} \partial_t^2) \beta_i(k, \omega) [\partial_t^2 f(k, \omega, t)|_{t=\omega+\omega_{s0}} - \partial_t^2 f(k, \omega, t)|_{t=\omega-\omega_{s0}}] \\ &\quad + (2\omega'_{s0} \mathfrak{z}'_s + \omega''_{s0} \mathfrak{z}_s) \beta_i(k, \omega) [\partial_t f(k, \omega, t)|_{t=\omega+\omega_{s0}} + \partial_t f(k, \omega, t)|_{t=\omega-\omega_{s0}}]]. \end{aligned} \quad (4.13)$$

Using these generic results, we perform the integrals numerically. We do not encounter any additional problem and we find all the integrals safe in the infrared. Here too we have to subtract the leading-order terms for ultraviolet convergence and, putting all contributions together, the numerical integrations yield the result:

$$\omega_s(p) = m_s [(1 + 0.258824 e + \mathcal{O}(e^2)) + (0.5 + 0.056684 e + \mathcal{O}(e^2)) \bar{p}^2 + \mathcal{O}(\bar{p}^4)]. \quad (4.14)$$

This is exactly the same result (4.8) obtained with the late momentum expansion. We should note the fact that there is no infrared sensitivity in the above expression, even though it has been technically more involved to derive than the damping rate $\gamma_s(p)$.

5. Discussion

This work aimed at calculating the damping rate $\gamma_s(p)$ and energy $\omega_s(p)$ for scalars in the context of next-to-leading order hard-thermal-loop perturbation theory of scalar QED. For each of the two quantities, we have carried out the calculation in two ways: using both a late

and early external momentum expansion, respectively after and before the Matsubara sum is done and the analytic continuation to real energies taken. Though technically different, both methods yield the same results for the damping rate and energy.

As mentioned in the introduction, this issue is particularly interesting in view of an early work [35, 36] in which the early momentum expansion was used in the context of next-to-leading order hard-thermal-loop perturbation of hot QCD in order to extract analytically a value for the non-moving longitudinal-gluon damping rate $\gamma_{gl}(0)$. The very definition of that quantity imposed a systematic expansion in powers of the external momentum p , at least to second order. From a technical point of view, the question was where to perform the expansion. Ideally, the latest possible, at least after the Matsubara sum and analytic continuation to real energies are done. But technically that was unfortunately not feasible in QCD because of the complication of the intermediary steps, at least if one wanted to carry through analytically. The early momentum expansion was used instead, and legitimately the occurrence of infrared sensitivity of the result was partially incriminated on this particular step.

The present work shows that infrared divergences can occur without the early momentum expansion. This is clear in the transverse contribution $\gamma_{st}(p)$ to the scalar damping rate, see (3.16). Remember that up to order p^2 , this result with the logarithmic divergence is already found in [40]. The reason we have pushed the calculation to order p^4 is to demonstrate that other forms of divergences do occur, here a $1/\eta^2$. This was the case in QCD too. Since these results are obtained without the early momentum expansion and the same results are obtained with it, we think we have here a strong indication that this latter is not responsible for any infrared sensitivity.

Furthermore, infrared sensitivity is not systematic when we perform a momentum expansion, late or early. This is exemplified in the longitudinal contribution $\gamma_{sl}(p)$ to the damping rate, see (3.16), and, maybe more pertinently since the calculations are more intricate, in the scalar energy $\omega_s(p)$. It is true that only fourth order in p is included in the determination of $\gamma_{sl}(p)$ and second order in the determination of $\omega_s(p)$, but the trend of the calculations indicates that higher-order coefficients will eventually be safe.

A couple of objections may still rise. First, one could argue that in fact, the late and early momentum expansions are actually the same for the quantities we have treated since the expressions of these, before any momentum expansion is performed, are still not analytically closed in terms of p . To answer this objection, one can go to the photonic sector of scalar QED treated in [13]. Indeed, the HTL-summed next-to-leading order photon self-energies are evaluated in closed form for all ω and p . Let us focus only on the longitudinal photons. Three regions in ω and p are to be distinguished: $\omega^2 < p^2$ (region I), $p^2 < \omega^2 < 4m_s^2 + p^2$ (region II) and $4m_s^2 + p^2 < \omega^2$ (region III). The longitudinal next-to-leading order HTL-summed photon self-energy is found to have the following expression [13]:

$$\delta^* \Pi_l(\omega, p) = \frac{e^2 T}{8\pi} \frac{\omega^2 - p^2}{p^2} \left[4m_s + 2i\varepsilon - i \frac{\omega^2}{p} \ln \left(\frac{2m_s - i(\varepsilon + p)}{2m_s - i(\varepsilon - p)} \right) \right], \quad (5.1)$$

where the prefix δ indicates here too that the known leading hard-thermal-loop contribution has been subtracted for ultraviolet convergence, and $\varepsilon = \Theta$ in regions I and III and $\varepsilon = i|\Theta|$

in region II, with $\Theta^2(\omega, p) = \omega^2(\omega^2 - p^2 - 4m_s^2) / (\omega^2 - p^2)$. The full longitudinal-photon dispersion relation is:

$$\Omega_l^2 = p^2 + \Pi_l(\Omega_l, p), \tag{5.2}$$

where Π_l is the full longitudinal-photon self-energy, and, up to next-to-leading order, writes:

$$\Omega_l^2(p) = \omega_l^2(p) + \frac{\delta^* \Pi_l(\omega_l, p)}{1 - \partial_{\omega_l^2} \delta \Pi_l(\omega_l, p)}, \tag{5.3}$$

where $\delta \Pi_l(\omega, p)$ is the hard thermal loop given by [13]:

$$\delta \Pi_l(\omega, p) = 3m_p^2 \left(1 - \frac{\omega^2}{p^2}\right) \left(1 - \frac{\omega}{2p} \ln \frac{\omega + p}{\omega - p}\right), \tag{5.4}$$

with $m_p = eT/3$ the photon thermal mass and $\omega_l(p)$ the on-shell longitudinal photon energy, solution to (2.8) to lowest order where only the hard thermal loop is kept in the self-energy. Using (5.4), we can rewrite (2.9) as:

$$\Omega_l^2(p) = p^2 + \delta \Pi_l(\omega_l, p) + \frac{2\omega_l^2(p)}{3m_p^2 + p^2 - \omega_l^2(p)} \delta^* \Pi_l(\omega_l, p). \tag{5.5}$$

Remember that all these results involve no expansion in p and the Matsubara sum and analytic continuation to real energies are already done. Also, all intermediary integrals are performed. Now we expand. The region of interest to us is region II, where we are allowed to perform the following expansion:

$$\omega_l(p) = 1 + \frac{3}{10} \tilde{p}^2 - \frac{3}{280} \tilde{p}^4 + \mathcal{O}(\tilde{p}^6); \quad \tilde{p} = p/m_p. \tag{5.6}$$

Using this, we perform the expansion of (2.10) in powers of \tilde{p} . It is straightforward and we find:

$$\Omega_l^2(p) = m_p^2 \left[(1 - 0.368e + \mathcal{O}(e^2)) + \left(\frac{3}{5} - 0.0536e + \mathcal{O}(e^2) \right) \tilde{p}^2 + \mathcal{O}(\tilde{p}^4) \right]. \tag{5.7}$$

Here we have together the leading and next-to-leading orders in the coupling e . Note that there is no infrared sensitivity. What remains now is to perform the expansion in powers of \tilde{p} before the Matsubara sum and analytic continuation to real energies are done, and carry on with steps similar to what we have gone through in this work. This we do and we obtain exactly the *same* result (5.7). There is no need to display the results we think. This is an additional indication that the early momentum expansion works. Of course, when it can be avoided like in the present context of scalar QED, there is no need to use it. But in the context of hot QCD, a more realistic theory, it is sometimes necessary if one wants to manipulate analytically.

One may then argue that if the calculations are to lead to infrared sensitivity, then the regularization with an infrared cut-off η should not be used in the first place. To answer this, one can say that, first of all, infrared sensitivity is not systematic in all similar quantities that are treated by similar methods. We have seen a good example of this in

this work, which is the two contributions to the scalar damping rate $\gamma_s(p)$: the longitudinal contribution $\gamma_{sl}(p)$ is infrared safe and the transverse contribution $\gamma_{st}(p)$ infrared sensitive, though both are calculated in exactly the same manner. Hence infrared sensitivity may be a feature of the physical quantity in question. Furthermore, we could have taken the transverse contribution $\gamma_{st}(p)$, made no expansion in p and tried to determine it numerically for different values of p without introducing any infrared cutoff. Then the internal momentum k -integration would not converge, because precisely of the very infrared sensitivity. One could argue that the regularization should be different from the naive introduction of an infrared cut-off. But, in whatever way it is chosen, regularization will only exhibit a potential infrared sensitivity; it will not remove or cancel it. For that purpose, some other procedure has to be invoked, and we have never meant to do that. This particular issue would be pursued elsewhere.

Finally, one may legitimately ask how one circles back from this work to the motivating issue itself, namely the determination of the longitudinal-gluon damping rate $\gamma_{gl}(0)$. The introduction of an infrared cutoff η means that a given quantity λ like the damping rate or the energy is the sum of two contributions:

$$\lambda = \lambda_{0-\eta} + \lambda_{\eta-\infty}, \tag{5.8}$$

where $0-\eta$ means integration over loop momenta from zero to η and $\eta-\infty$ means integration from η to ∞ . All our calculations determine only $\lambda_{\eta-\infty}$ and not $\lambda_{0-\eta}$. Two distinct situations may arise: (i) The limit $\eta \rightarrow 0$ of $\lambda_{\eta-\infty}$ is finite, in which case $\lambda_{0-\eta}$ goes to zero in this limit and the whole λ is safely reached via $\lambda_{\eta-\infty}$ in the limit $\eta = 0$. This is the case for $\gamma_{sl}(p)$ and $\omega_s(p)$ in the present work. (ii) The limit $\eta \rightarrow 0$ of $\lambda_{\eta-\infty}$ is divergent, which means $\lambda_{0-\eta}$ is of significant contribution that *must not* be neglected. This is the situation of $\gamma_{st}(p)$ in the present work and, indeed, of $\gamma_{gl}(0)$ in [35, 36]. In this second case, the physical λ cannot be reached via $\lambda_{\eta-\infty}$ in the limit $\eta \rightarrow 0$. This is what could be called infrared or ultrasoft sensitivity.

In the work [39] has been derived a Slavnov-Taylor identity for the gluon polarization tensor in the Coulomb gauge and applied to the next-to-leading-order contribution. The identity recovers already known transversality conditions, and when applied to the imaginary part of next-to-leading order HTL-summed gluon self-energy, it implies the equality between the damping rates for non-moving transverse and longitudinal gluons. It is instructive to recall the main steps of [39]. The Slavnov-Taylor identity is:

$$P^\nu \Pi_{\mu\nu}^{(g)}(P) = - \left[\delta_\mu^\nu P^2 - P_\mu P^\nu + \Pi_\mu^{(g)\nu}(P) \right] \Pi_{g\nu}^{(g)}(P), \tag{5.9}$$

where $\Pi_{\mu\nu}^{(g)}(P)$ is the full gluon self-energy and $\Pi_{g\nu}^{(g)}(P)$ is related to the Coulomb-ghost self-energy $\Pi_g^{(g)}$ via the relation $\Pi_g^{(g)}(P) = p^i \Pi_{gi}^{(g)}(P)$. Note that identity (5.9) is exact and valid for all momenta P . In the sequel, concentrate on the soft momentum region. When reading the identity (5.9) in perturbation in powers of the coupling g , we can deduce the two already known transversality conditions:

$$P^\nu \delta \Pi_{\mu\nu}^{(g)}(P) = 0; \quad P^\mu P^\nu \ast \Pi_{\mu\nu}^{(g)}(P) = 0, \tag{5.10}$$

where $\delta\Pi$ is the hard thermal loop and $^*\Pi$ the HTL-dressed one-loop-order self-energy. For the present purposes, we need the Coulomb-ghost self-energy only to lowest order, and it is derived in [39] to read:

$$\Pi_{g\mu}^{(g)}(P) = -\frac{g^2 N_c T}{16} \delta_\mu^i \hat{p}^i, \quad (5.11)$$

which is already of order $g^2 T$ and has the limit $\Pi_{g\mu}^{(g)}(\omega, \mathbf{p} \rightarrow \mathbf{0}) = 0$. From this we get the relation [39]:

$$P^\nu {}^*\Pi_{\nu\mu}^{(g)}(P) = \frac{g^2 N_c T}{16} \frac{\omega}{p} {}^*\Delta_{gl}^{-1}(P) \left(1, \frac{\omega}{p} \hat{p}\right)_\mu, \quad (5.12)$$

where ${}^*\Delta_{gl}$ is the dressed longitudinal gluon propagator. Since the right-hand side of (5.12) is real, we immediately obtain:

$$P^\nu \text{Im} {}^*\Pi_{\nu\mu}^{(g)}(P) = 0. \quad (5.13)$$

From here, it is straightforward to show that [39]:

$$\gamma_{gl}(0) = \gamma_{gt}(0), \quad (5.14)$$

with the same definition of the damping rates as the one adopted in [35, 36].

The above arguments are correct and form an important confirmation that, on physical grounds, the longitudinal and transverse gluon damping rates have to be equal at zero momentum. We have emphasized on many occasions that any consistent computational scheme ought to yield this equality. However, the calculation in [35, 36] and similar works are explicit whereas those in [39] are formal: the loop integrals and Matsubara sums are never done explicitly; no infrared cutoff is introduced to regulate the infrared sector and there is no mention of what region should be kept in the integral/sum. All these issues are at the heart of any explicit calculation and have to be dealt with, often case by case: when sensitivity is expected, different sets of rules can and do lead to different results.

Having found an infrared sensitivity in $\gamma_{gl}(0)_{\eta \rightarrow \infty}$ means we must not infer that the whole $\gamma_{gl}(0)$ is the limit $\eta \rightarrow 0$ of the quantity $\gamma_{gl}(0)_{\eta \rightarrow \infty}$ we calculated. Now the identity (5.14) applies to the *whole* $\gamma_{gl}(0)$ and *not* to $\gamma_{gl}(0)_{\eta \rightarrow \infty}$. From this standpoint, this identity does not invalidate the finding of [35, 36] but confirms formally the importance of the ultrasoft contributions $\gamma_{gl}(0)_{0-\eta}$ which we have not calculated. We want to stress that the result of [35, 36] has never been used to imply a difference between $\gamma_{gl}(0)$ and $\gamma_{gt}(0)$: on the contrary, relying on the physical equality between the two, the result has been used instead to argue for the importance of the ultrasoft region and the sensitivity of the hard-thermal-loop perturbation to it.

Infrared cutoff regularization has been used as a mean to carry forward with the calculation beyond leading order. Ideally, one would have preferred to use a regularization scheme that allows k to run from 0 to ∞ . The introduction by hand of a magnetic mass in the static magnetic propagator is a possibility, but that has been questioned in the past: it is not clear that the chromomagnetic effects would manifest themselves only in the form of a simple shift in the static chromomagnetic propagator. This regularization scheme may

also have problems with gauge dependence. In any case, even that scheme is not free from infrared divergences.

One may suggest to use dimensional regularization and there are already explicit expressions for the dressed propagators in d dimensions, see for example [43, 44]. But it does not seem a priori straightforward to work with on-shell expressions involving both dressed propagators and vertices, expressions of the generic type

$$T \sum_{k_0} \int d^d k \int d\Omega_{S_1} \int d\Omega_{S_2} \frac{f(\hat{s}_1, \hat{s}_2)}{P_{S_1} K_{S_1} P_{S_2} K_{S_2}} * \Delta(K) * \Delta(Q),$$

knowing that p has to be kept non-zero, even in the calculation of $\gamma_{gl}(0)$. And even in the eventuality of these integrals being done, one is not assured of being immune from infrared sensitivity. Indeed, consider as an example the result in [43] for the dimensionally regulated hard gauge boson damping rate to next-to-leading log order:

$$\gamma = \frac{g^2 N_c T}{4\pi} [-1/\epsilon + \ln(m/\bar{M})], \quad (5.15)$$

where N_c the number of colors, m the Debye screening mass, \bar{M} the $\overline{\text{MS}}$ renormalization scale set at the order of γ itself and $\epsilon = 3 - d$. An interplay between the infrared and ultraviolet divergences occurring in the calculations as explained in [43] removes the infrared sensitivity of this coefficient to recover the known result estimated by other means [31], but the point to retain is that the dimensionally-regulated calculation itself is sensitive to the regularization scheme. One can argue that matters will be different for the non-moving plasmon damping rate $\gamma_{gl}(0)$ since, on physical grounds and by gauge invariance, one ought to get a clean finite number. This is what one hopes for of course, but that remains to be seen explicitly.

A. The other terms in late momentum expansion

This appendix gives details on how we perform the integrals involved in the late momentum expansion of the damping rate $\gamma_s(p)$ we have not shown in subsection 3.2. We will try to use as clear and concise a notation as possible.

The second type of integrals we have to deal with is the following:

$$I_{\rho \partial_k \rho} = \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \rho_i(\omega, k) \partial_k \rho_s(1 - \omega, k). \quad (\text{A.1})$$

As before, the discussion has to be carried out contribution by contribution, using the structure of the spectral functions (3.12) and (3.15). The first contribution to consider is the one that involves two delta functions

$$\int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \mathfrak{z}_i f(k, \omega, 1 - \omega) \delta(\omega \mp \omega_i) \partial_k [\mathfrak{z}_s \delta(1 - \omega \mp \omega_{s0})].$$

When the derivative over k is applied to $\mathfrak{z}_s(k)$, we have zero contribution. Indeed, in order for it to be nonvanishing, the energies must here too satisfy $\pm \omega_{s0}(k) \pm \omega_i(k) = 1$,

forbidden by kinematics as already mentioned. But when applying the k -derivative to the delta function, we also have zero contribution. This is simply because there is no intersection between the supports of the two delta functions, even if they are involved through derivatives, first order as in here or higher. The other contribution is a cut-residue, namely

$$\int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \beta_i(k, \omega) \Theta(k - |\omega|) \partial_k [\mathfrak{z}_s \delta(1 - \omega \mp \omega_{s0})].$$

We first apply the derivative over k to \mathfrak{z}_s and we get the piece

$$\int_{\eta}^{+\infty} dk \mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_i(k, 1 - \omega_{s0}),$$

where \mathfrak{z}'_s stands for $d\mathfrak{z}_s(k)/dk$. We then apply it to $\delta(1 - \omega \mp \omega_{s0})$ and use the standard rules regulating the handling of the delta distribution and we always check the results by regularizing either the derivative ∂_k or the delta function itself. Only $\omega = 1 - \omega_{s0}$ contributes and we obtain

$$\int_{\eta}^{+\infty} dk \omega'_{s0} \mathfrak{z}_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)]|_{\omega=\omega_{s0}}.$$

Putting the above two contributions together, we obtain the following result:

$$I_{\rho\partial_k\rho} = \int_{\eta}^{+\infty} dk [\mathfrak{z}'_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_i(k, 1 - \omega_{s0}) + \omega'_{s0} \mathfrak{z}_s \partial_{\omega} [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)]|_{\omega=\omega_{s0}}]. \quad (\text{A.2})$$

The third type of integrals we have to deal with is one that involves a second derivative in k :

$$I_{\rho\partial_k^2\rho} = \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \rho_i(k, \omega) \partial_k^2 \rho_s(1 - \omega, k). \quad (\text{A.3})$$

The steps to treat the different contributions parallel those followed for $I_{\rho\partial_k\rho}$. As explained before, the $\delta\delta$ contribution is zero because of kinematics. Therefore, the only contribution to look at is

$$\int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \beta_i(k, \omega) \Theta(k - |\omega|) \partial_k^2 [\mathfrak{z}_s \delta(1 - \omega \mp \omega_{s0})].$$

We have to use the identity $\partial_k^2 (\mathfrak{z}_s \delta) = \mathfrak{z}''_s \delta + 2\mathfrak{z}'_s \partial_k \delta + \mathfrak{z}_s \partial_k^2 \delta$, where \mathfrak{z}''_s stands for the second derivative of $\mathfrak{z}_s(k)$. The term involving $\mathfrak{z}''_s \delta$ yields $\int_{\eta}^{+\infty} dk \mathfrak{z}''_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_i(k, 1 - \omega_{s0})$. The two other terms $\mathfrak{z}'_s \partial_k \delta + \mathfrak{z}_s \partial_k^2 \delta$ give together the following result

$$\int_{\eta}^{+\infty} dk [(\mathfrak{z}_s \omega''_{s0} + 2\mathfrak{z}'_s \omega'_{s0}) \partial_{\omega} + \mathfrak{z}_s \omega_{s0}^2 \partial_{\omega}^2] [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)]|_{\omega=\omega_{s0}}.$$

The procedure is to replace ∂_k by $\omega_{s0} \partial_{\omega}$ and apply the usual rules regulating the handling of the delta distribution. Putting the above results together, we obtain:

$$I_{\rho\partial_k^2\rho} = \int_{\eta}^{+\infty} dk [\mathfrak{z}''_s f(k, 1 - \omega_{s0}, \omega_{s0}) \beta_i(k, 1 - \omega_{s0})$$

$$+ [(\mathfrak{z}_s \omega''_{s0} + 2\mathfrak{z}'_s \omega'_{s0}) \partial_\omega + \mathfrak{z}_s \omega'^2_{s0} \partial_\omega^2] [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)] \Big|_{\omega=\omega_{s0}}. \quad (\text{A.4})$$

Along similar lines we find the integral involving the third derivative in k :

$$\begin{aligned} I_{\rho \partial_k^3 \rho} &= \int_\eta^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \rho_i(\omega, k) \partial_k^3 \rho_s(1 - \omega, k) \\ &= \int_\eta^{+\infty} dk \left[\left[\mathfrak{z}_s^{(3)} + \left(3\mathfrak{z}''_s \omega'_{s0} + 3\mathfrak{z}'_s \omega''_{s0} + \mathfrak{z}_s \omega_{s0}^{(3)} \right) \partial_\omega \right. \right. \\ &\quad \left. \left. + \left(3\mathfrak{z}'_{s0} \omega'^2_{s0} + 3\mathfrak{z}_s \omega''_{s0} \omega'_{s0} \right) \partial_\omega^2 + \mathfrak{z}_s \omega_{s0}^3 \partial_\omega^3 \right] [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)] \Big|_{\omega=\omega_{s0}} \right], \end{aligned} \quad (\text{A.5})$$

and the one involving the fourth derivative:

$$\begin{aligned} I_{\rho \partial_k^4 \rho} &= \int_\eta^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega, 1 - \omega) \rho_i(\omega, k) \partial_k^4 \rho_s(1 - \omega, k) \\ &= \int_\eta^{+\infty} dk \left[\left[\mathfrak{z}_s^{(4)} + \left(4\mathfrak{z}_s^{(3)} \omega'_{s0} + 6\mathfrak{z}''_s \omega''_{s0} + 4\mathfrak{z}'_s \omega_{s0}^{(3)} + \mathfrak{z}_s \omega_{s0}^{(4)} \right) \partial_\omega \right. \right. \\ &\quad \left. \left. + \left(6\mathfrak{z}''_s \omega'^2_{s0} + 12\mathfrak{z}'_s \omega''_{s0} \omega'_{s0} + \mathfrak{z}_s \left(4\omega_{s0}^{(3)} \omega'_{s0} + 3\omega_{s0}''^2 \right) \right) \partial_\omega^2 + \left(4\mathfrak{z}'_s \omega_{s0}^3 + 6\mathfrak{z}_s \omega''_{s0} \omega'^2_{s0} \right) \partial_\omega^3 \right. \right. \\ &\quad \left. \left. + \mathfrak{z}_s \omega_{s0}^4 \partial_\omega^4 \right] [f(k, 1 - \omega, \omega) \beta_i(k, 1 - \omega)] \Big|_{\omega=\omega_{s0}} \right]. \end{aligned} \quad (\text{A.6})$$

The next types of integrals we must handle are those coming from deriving the spectral density $\rho_s(\omega_{s0}(p) - \omega, k)$ in (3.20) and (3.21) with respect to p . It is convenient to handle these terms by reintroducing $\delta(\omega_{s0}(p) - \omega - \omega')$ and performing the derivative on it. The first such generic term is:

$$\begin{aligned} I_{\rho \rho \partial_\omega \delta} &= \int_\eta^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \partial_\omega \delta(1 - \omega - \omega') f(k, \omega, \omega') \rho_i(\omega, k) \rho_s(\omega', k) \\ &= - \int_\eta^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \partial_\omega [f(k, \omega, \omega') \rho_i(k, \omega)] \Big|_{\omega'=1-\omega} \rho_s(1 - \omega, k). \end{aligned} \quad (\text{A.7})$$

Here too the residue-residue contribution is zero because of the same kinematics. What remains to calculate is the cut-residue contribution, which is equal to

$$- \int_\eta^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \mathfrak{z}_s \partial_\omega [f(k, \omega, \omega') \beta_i(k, \omega) \Theta(k - |\omega|)] \Big|_{\omega'=1-\omega} \delta(1 - \omega \mp \omega_{s0})$$

and straightforwardly shown to yield:

$$I_{\rho \rho \partial_\omega \delta} = - \int_\eta^{+\infty} dk \mathfrak{z}_s \partial_\omega [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)] \Big|_{\omega=1-\omega_{s0}}. \quad (\text{A.8})$$

The other term involves a second order ω -derivative; it is worked out straightforwardly:

$$\begin{aligned} I_{\rho \rho \partial_\omega^2 \delta} &= \int_\eta^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \partial_\omega^2 \delta(1 - \omega - \omega') f(k, \omega, \omega') \rho_i(\omega, k) \rho_s(\omega', k) \\ &= - \int_\eta^{+\infty} dk \mathfrak{z}_s \partial_\omega^2 [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)] \Big|_{\omega=1-\omega_{s0}}. \end{aligned} \quad (\text{A.9})$$

Last are integrals involving an ω -derivative over $\delta(1 - \omega - \omega')$ and k -derivatives over ρ_s . The treatment is similar and straightforward too. With one k -derivative we obtain:

$$\begin{aligned}
 I_{\rho\partial_k\rho\partial_\omega\delta} &= \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \partial_\omega \delta(1 - \omega - \omega') f(k, \omega, \omega') \rho_i(\omega, k) \partial_k \rho_s(\omega', k) \\
 &= - \int_{\eta}^{+\infty} dk [\mathfrak{z}'_s \partial_\omega [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)]|_{\omega=1-\omega_{s0}} \\
 &\quad - \mathfrak{z}'_s \omega'_{s0} \partial_{\omega'} [\partial_\omega [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)]|_{\omega=1-\omega'}]|_{\omega'=\omega_{s0}}, \tag{A.10}
 \end{aligned}$$

and with a second-order k -derivative we get:

$$\begin{aligned}
 I_{\rho\partial_k^2\rho\partial_\omega\delta} &= \int_{\eta}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \partial_\omega \delta(1 - \omega - \omega') f(k, \omega, \omega') \rho_i(\omega, k) \partial_k^2 \rho_s(\omega', k) \\
 &= - \int_{\eta}^{+\infty} dk [\mathfrak{z}''_s \partial_\omega [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)]|_{\omega=1-\omega_{s0}} \\
 &\quad - 2\mathfrak{z}'_s \omega'_{s0} \partial_{\omega'} [\partial_\omega [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)]|_{\omega=1-\omega'}]|_{\omega'=\omega_{s0}} \\
 &\quad - \mathfrak{z}_s (\omega''_{s0} \partial_{\omega'} + \omega'^2_{s0} \partial_{\omega'}^2) [\partial_\omega [f(k, \omega, \omega_{s0}) \beta_i(k, \omega)]|_{\omega=1-\omega'}]|_{\omega'=\omega_{s0}}]. \tag{A.11}
 \end{aligned}$$

These are generically all the types of integrals we will have to evaluate. In each case, we have to examine the infrared behavior, i.e., the behavior close to η . In the eventuality of the presence of an infrared divergence, we have to extract it analytically. The remaining finite part of the integral is generally performed numerically.

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